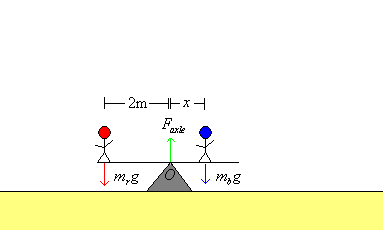
**Example: See-saw equilibrium**

Suppose red (m = 70kg) is standing on a relatively massless see-saw 2m from the pivot, and blue (m = 90kg) is standing on the other side. Where is blue standing if the see-saw is still, and what is the force that the pivot exerts on the see-saw? Assume the axle is frictionless (so τaxle = 0).

We start by labelling all the forces acting on the see-saw.



Now take our system to be the see-saw. Since it isn’t going anywhere, we must have,



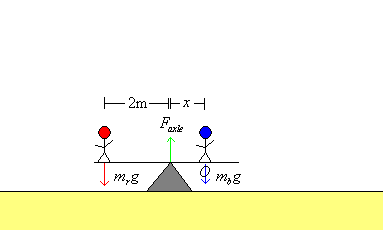
Since it isn’t rotating either, we must have that,



It is up to us to choose an appropriate reference point, O, about which to add up the torques. A natural choice would be the pivot, but this isn’t necessary. Because the see-saw isn’t rotating about the red guy, or the blue guy, etc., the net torque about their positions must also be 0. So we could put O at their locations too. Nonetheless, we’ll use the pivot as O this time,



In order to emphasize that we needn’t set O to be the pivot, let’s set it elsewhere and see that we get the same result. We’ll put it at the location of the blue person.





So hopefully this illustrates the point that you can put the reference point wherever you want.

Another point to consider is the following. You can use the torque equation as many times as you want, in lieu of the force equation. **Often times, using the torque equation is simpler than using the force equation**.

For instance we could’ve solved the problem this way. We won’t use the force equation at all. We’ll just state that the see-saw isn’t rotating about the pivot, which implies that (putting O at the pivot)

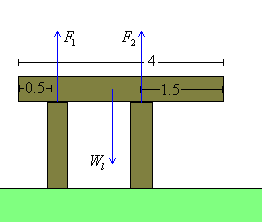


as we know. And then to get Faxle, we’ll use the fact that the see-saw isn’t rotating about the blue guy either (or we could use the red one)



**Example: Egyptian temple**

Consider that you’re building a temple. So you put two posts in the ground and place the lintel beam (m = 5000 kg) across them, arranged as follows, with the lintel cantilevered over the right post. What are the forces F1 and F2 that the posts exert on the lintel?



First we label the forces. F1 and F2 are obvious. Wℓ is the weight of the lintel acting at its center of mass (the middle of the lintel assuming its uniform). We can use N2L for translation and rotation to determine this. First, let’s use N2L for rotation. To simplify the analysis, let’s use as our axis of rotation the point O at the location of F1. Then,

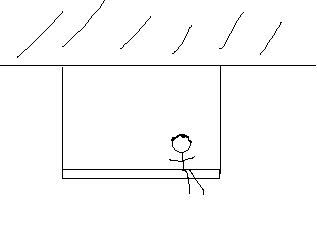


and now we’ll use N2L for translation to determine F1. We have,



**Problem 5.**

A child sits on a hammock, as shown below. What is the tension in the two ropes? Suppose the child has a mass of 35kg, and the hammock a mass of 45kg. Also, let the hammock be 2m long and let the child be sitting 0.5m from the right end.



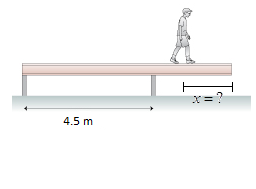
Let F1 be tension in left rope, and F2 be tension in right rope. Let our axis of rotation be where left rope connects to hammock. Then torque equation reads:



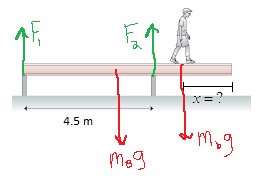
Then from N2L in y-direction we have:



**Question 3**. A 50kg, 7m long beam is supported, but not attached to, the two posts in the figure. A 25kg boy starts walking along the beam. How close can he get to the end before the the beam tips over?



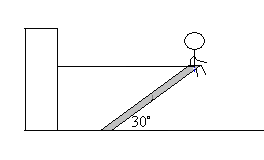
Forces look like this:

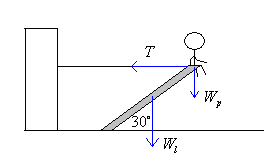


When it’s about to tip, F1 = 0. So doing N2L about the F2 reference point we get:



4. Suppose that the 2m long board below is pinned to the ground, that it weighs 100N, and a 300N child is sitting on its edge. Calculate the value of the tension, T, in the rope necessary to maintain the board in static equilibrium. You may assume gravity acts through the center of the board.



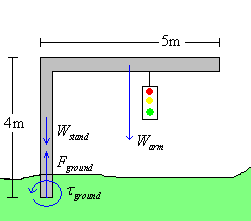


Apply the torque equation about the base of the ladder. And we’ll have,



**Example: Traffic light**

Consider a traffic light stand. Suppose the stand has a mass of 300kg, and height 4m, while the cross arm has a mass of 200kg and length of 5m.. What torque must the ground exert on the base to keep the traffic light stand from falling over? What force must the ground exert on the stand?



Note that the ground kind of acts like an axle. It exerts both a force on the base of the traffic light stand upwards, Fground, and a CCW torque, τground on it. Taking the base of the cross-arm as our axis of rotation, we have,

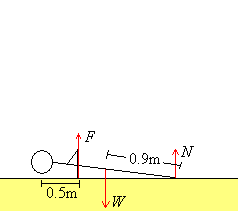


Using N2L we have



**Example: Force required to do a push-up**

Suppose you are 1.8m tall. Further suppose your center of mass is close to your center. And we’ll suppose your shoulders to be about 0.5m from the top of your head. If you’re in the push-up position then, what force do you exert as a fraction of your weight to push yourself up? What force does the ground exert on your toes? (Assume the person is almost completely flat).



Let O be at the person’s feet. Then according to N2L for rotation we have,



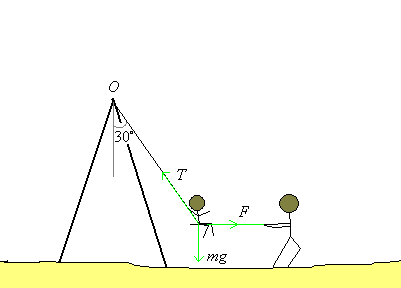
The force your arms would have to exert is app. 70% of your weight (105 lbs, if you weigh 150lbs.). Using N2L for translation, we can determine N.



so N would be about a third of your weight.

**Example: Holding a child on a swing**

This example illustrates what is useful about looking at things from a rotational perspective. Suppose you have someone on a swing. Suppose the person weighs 200N, the length of the swing is ℓ = 2m. And you’re holding it at an angle of 30 degrees with respect to the vertical. What force, F, is required?



Such a problem we can solve using N2L and adding up the forces, setting them equal to 0. This would give us 2 equations and 2 unknowns, T, and F. The nice thing about the rotational version of N2L is that we can solve directly for F, without needing T first. For instance, let O be the top of the swing, where the rope connects to the support. And then let’s add the torques on the swing about that point,

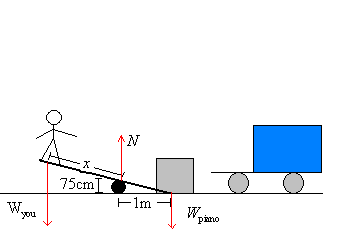


The torque exerted by T is 0, since no part of it is perpendicular to its lever arm, ℓ. Thus T drops out of the equation, and we automatically eliminate one unknown! Proceeding,



**Example: Force multiplication via lever**

Suppose you (m = 70kg) want to lift a 200kg (~ 450 lbs) piano onto the bed of a truck. So you put use a sturdy board over a fulcrum (height of fulcrum is 75cm) one meter away from the piano. How far away from the fulcrum do you need to stand?



In order to lift the piano, we must have that the torque your weight exerts on the lever is at least equal to the torque on the lever exerted by the weight of the piano. So we must have, (because we want the lever to rotate CCW)



Now ryou = x, Wyou = (70)(9.8) = 686N, and φyou is the angle between x and Wyou. This is

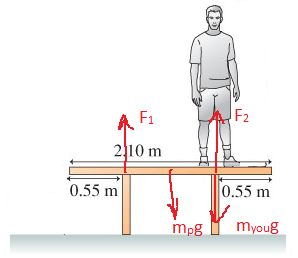
φ = tan-1(0.75/1) = 37˚. Next, rpiano can be obtained from the Pythagorean theorem,

. Wpiano = (200)(9.8) = 1960N. Finally φpiano = 37˚ as well. Filling all these in,



So to lift the piano, you’d have to stand 3.57m from the fulcrum.

**Question 4**. Suppose you are standing on a plank lying on top of two posts. If the plank has a mass m = 15kg, you have a mass m = 70kg, and you are standing directly on top of the right post, what force does each post exert on the plank?





Taking F2 as our axis of rotation, we have for the torque equation:

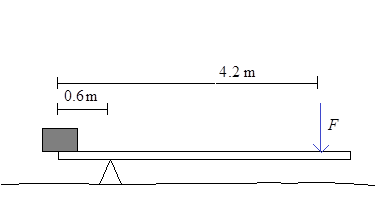


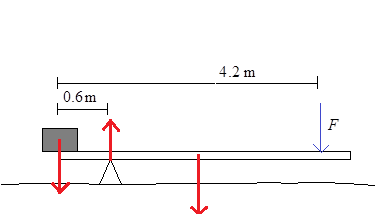
Filling this into the Fy equation we get:



**Problem 4.**

Suppose you’re holding up a large box (m = 325 kg) with a board as shown below. The length of the board is ℓ = 4.7m, and its mass is m = 13 kg. What force F must you apply? Don’t forget to take into account the mass of the board ☺



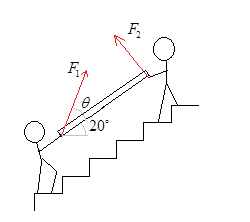


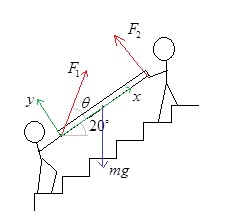
The forces acting on the board are shown above. Now we set our axis of rotation to be the fulcrum and apply the torque equation.



**Problem 2**

You and a friend are carrying a heavy board up the stairs. The board has a mass m = 150kg and is 2m long. Calculate the magnitude F1 and direction θ of force 1, and also the magnitude of force 2 (note force 2 is directed perpendicular to the board). Don’t forget to include the gravitational force, and remember it acts at the middle of the board. To do this problem, I would orient my axes so that x points along the board, and y points perpendicular to it. Then I would do the torque equation about the origin of **F**1, followed by N2L in the ‘x’ direction, and last N2L in the ‘y’ direction.





The torque equation is:



N2L in the x-direction reads:



N2L in the y-direction reads:



Now let’s solve for F1 in the top equation and plug it into the bottom equation:

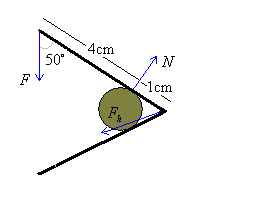


Plugging this into an F1 equation again:



**Example: Nut cracker**

As another example of how levers can multiply applied force, consider the nut-cracker shown below. If you exert a force of F on the ends as shown, what force is exerted on the nut? What force is exerted on the hinge? To determine this, we will consider the forces acting on the top lever itself. There is the force F pushing down, the force the nut exerts on the lever, N, and the force the hinge exerts on the lever, Fh. The forces acting on the bottom lever are not shown.



To determine N, let’s apply N2L for rotation to the upper lever, taking our axis of rotation to be the hinge. Then we have,



so we see that the force the nut exerts on the lever (which by N3L is the force the lever exerts on the nut) is 3.83 times the force applied to the lever. This is the usefulness of levers. Now what of the force that the hinge exerts on the lever (this would be important to know if we don’t want the hinge to break)? We can use N2L for translation to determine this.



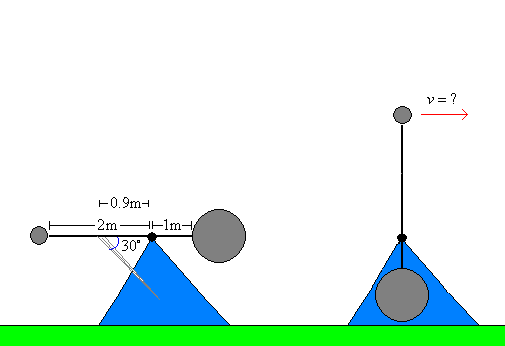
The magnitude and direction of **F**h is



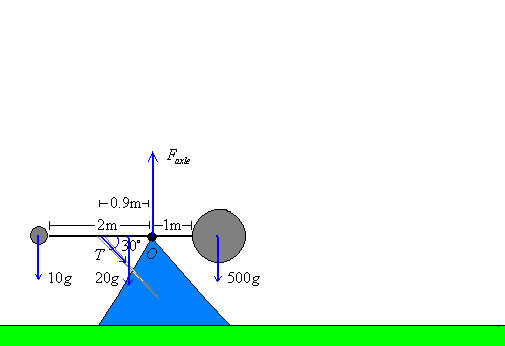
So the force on the hinge will also be much larger than the force applied to the lever, F.

**Example: Trebuchet**

Consider the trebuchet, a type of catapult looking device used during the middle ages to besiege enemy castle, and in modern times to hurl pumpkins and the like: Suppose the counterweight is 500kg, and the projectile is 10kg. And suppose that the catapult arm is uniform and has a mass of 20kg. What is the tension in the rope? What is the force of the pivot? Assume a frictionless axle.



To determine the tension we must draw in all of the forces acting on the lever arm. Gravity acts on the masses at the ends, the tension in the ropes acts 0.9m from the pivot, and the gravity acts on the catapult arm in the middle of the arm, since its uniform, which is 0.5m to the left of the pivot (distance not labelled). Finally, the axle exerts some force on the arm, Faxle, but by assumption, doesn’t exert a torque.

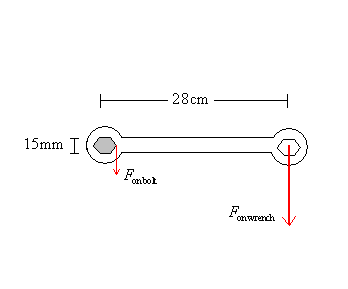


Now, to determine the tension T, we use the torque and force equations. It will be most advantageous to use the torque equation, putting O at the pivot, at the origin of the unknown force, Faxle. In that case we have,



**Problem 8.26**

The bolts on the cylinder head of an engine require tightening to a torque of 88 N·m. If a wrench is 28cm long, what force perpendicular to the wrench must the mechanic exert at its end? If the six-sided bolt head is 15mm in diameter, estimate the force applied near each of the six points by a socket wrench.



**Solution**

We’ll answer the last part first. We want the torque exerted on the bolts to be 88 Nm. This means that the torque supplied by the force Fon bolt must exert a torque of 88 Nm. In order for this to happen we must have (ignoring the sign)



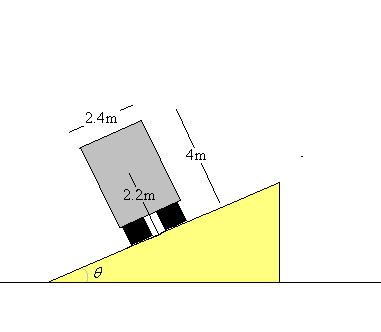
which is around 2600 lbs. of force! Now let’s consider the force that the person must exert on the wrench. We’ll reason this way. If the wrench exerts a torque of 88 Nm on the bolt, then according to Newton’s third law, the bolt must exert a torque of 88 Nm on the wrench. Now, in order for the wrench not to rotate, the person holding must exert an equal an opposite torque of 88 Nm on *it*. So we must have,



which is about 65 lbs. or so. So we can see the advantage a wrench offers. It would be much harder to exert a force of 11 730 lbs., than 65 lbs.

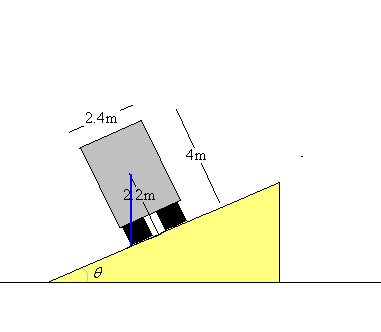
**Problem 9.63**

The center of gravity of a loaded truck depends on how the truck is packed. If it is 4.0m high and 2.4m wide, and its center of gravity is 2.2m above the ground (if it weren’t packed at all, the c.g. would just be halfway between the ground and the top of the truck), how steep a slope can the truck be parked on without the truck tipping over?

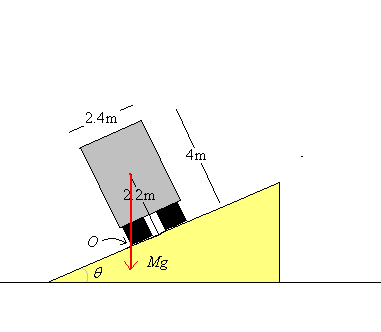


**Solution**

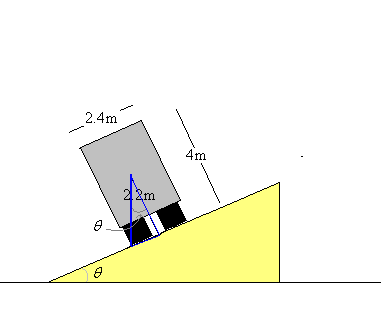
The maximum angle is that for which the c.g. is directly above the bottom left corner of left hand tire, illustrated below,



This is because the force of gravity acts along this line,



and if the angle were smaller, then gravity would exert a CW torque about O rotating the truck onto road. But if the angle were larger, than gravity would exert a CCW torque about O rotating the truck off of the road (you’ll have to draw these pictures yourself – I’m tired of using *MS paint* at this point ☺). This illustrates a general principle – that when the c.g. passes the outermost edge of an object, it will begin to fall. Now, to get the max angle, we will consider the triangle below,



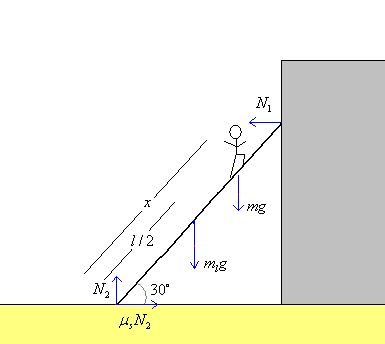
and note that the upper angle of the triangle is the same as the angle of the road – try to prove it to yourself. And thus we will have,



**Example: Ladder Problem**

Suppose a person is standing on a uniform ladder with mass (mℓ = 20kg) which is leaning against a wall. If the wall is frictionless, and the coefficient of static friction between the floor and the ladder is μs = 1.2, how high can the person (mp = 60kg) climb the ladder before it starts to slip?

First, we draw all the forces acting on the ladder. There is a normal force N1 that the wall exerts on the ladder. And a normal force N2 the floor exerts. Additionally, there is a friction force fs that the floor exerts on the ladder. We set this to its maximum value b/c we’re determining how high the person can go before the ladder slips (i.e., just before it starts to slip). Finally, the weight of the person acts on the ladder, and the weight of the ladder acts on the ladder. The ladder’s weight acts at the center of mass of the ladder, as discussed above, and since the ladder is uniform, this will be at the middle of the ladder.



So our unknowns are N1, N2, and x. To solve for them we can use the force equation and/or the torque equation. To start off, we’ll use the force equations,

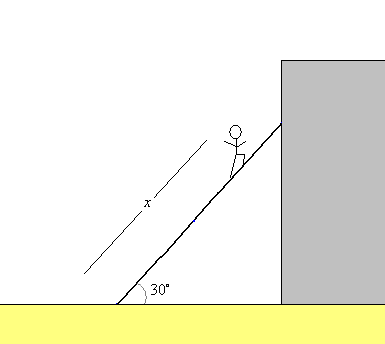


It is often convenient to use torque equation such that you put O at the location of one or more of the unknown forces. This will eliminate them from the equation, thereby simplifying the mathematics. To that end, we’ll put O at the bottom of the ladder and use the torque equation,

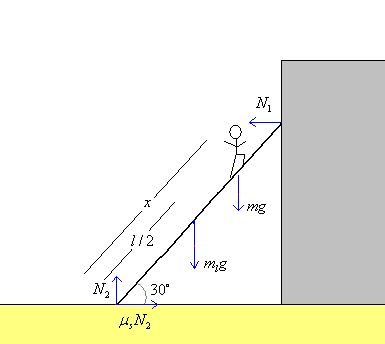


So the person can climb 8/10 the way up the ladder before it will start to slip.

**Question 4.** Suppose a person is standing on a uniform ladder with mass mℓ = 20kg and length ℓ = 7m which is leaning against a wall. If the wall is frictionless, and the coefficient of static friction between the floor and the ladder is μs = 1.2, how high can the person (mp = 60kg) climb the ladder before it starts to slip.



First, we draw all the forces acting on the ladder. There is a normal force N1 that the wall exerts on the ladder. And a normal force N2 the floor exerts. Additionally, there is a friction force fs that the floor exerts on the ladder. We set this to its maximum value b/c we’re determining how high the person can go before the ladder slips (i.e., just before it starts to slip). Finally, the weight of the person acts on the ladder, and the weight of the ladder acts on the ladder. The ladder’s weight acts at the center of mass of the ladder, as discussed above, and since the ladder is uniform, this will be at the middle of the ladder.



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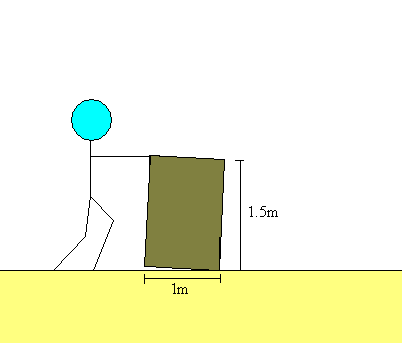
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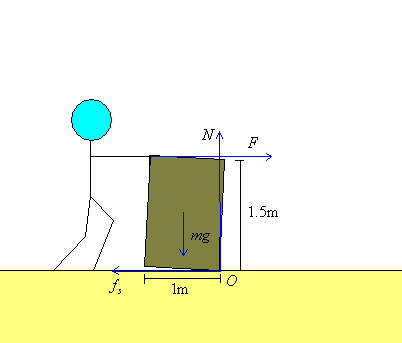
So the person can climb 0.8(7m) = 5.6m up the ladder before it will start to slip.

**Example: Pushing over a chest**

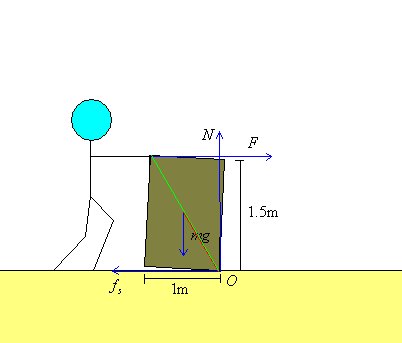
Suppose you have a 100kg chest that you want to push over, illustrated below. If you push on the top of the chest, horizontally, what force must you use to push it over?



First we label the forces acting on the chest. There is a normal force acting from the floor at the point of contact, as well as a friction force *fs*. Additionally, gravity will act on the chest. To accurately calculate the torque exerted by gravity we have to be more specific about where the gravitational force acts – turns out that it always acts at the ‘center of mass’ of the object, which usually coincides with the center of the object, when the mass is uniformly distributed. The forces are displayed below,



Now if we’re just about to push it over, then the forces and torques on the chest add up to zero (because it isn’t yet falling over). So we’ll add up the forces/torques and set to 0. For the torques, we’ll set our reference point at the lower right corner of the chest, since that will eliminate fs and N from the torque equations. And the lever arm for mg is labelled in red, and the lever arm for F is labelled in green.



Before we use the torque equation we need to know the angle between the F lever arm (green) and F and this is:



and the angle between the mg lever arm (red) and mg is:



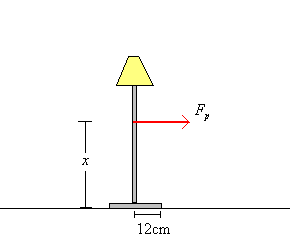
Then the torque equation is:



which is about 327/4.5 = 73 lbs.

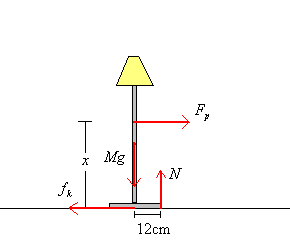
**Problem 9.28**

A person wants to push a lamp (mass m = 7.2kg) across the floor for which the coefficient of friction is 0.2. Calculate the maximum height, x, above the floor at which the person can push the lamp so that it slides rather than tips. Assume the center of mass of the lamp is 1m off of the floor.



**Solution**

First we must draw all the forces acting on the lamp. There is the force of gravity, the friction force, and the normal force from the floor. Now if we’re looking for the maximum height x before the lamp starts to the tip, then we may assume that the lamp is very nearly about to tip. If so, then it would tip about the right edge of its base, and this is where the normal force would act. So we will, in accordance with the problem statement, assume that the lamp is *just* about to tip – but isn’t quite.



And now we just write out our equations, assuming that the lamp is just about to tip/slide (therefore it isn’t moving yet and the acceleration, angular angular acceleration are 0).



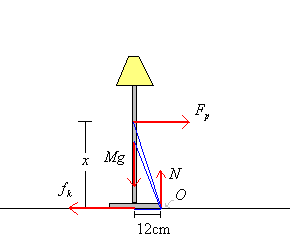
The y-equation reads,



The x-equation reads,



and now let’s apply the torque equation. Let the point where N acts be our axis of rotation, O. Then I’ll draw all of the lever arms in blue,



So writing out the torque equation,



Now the torque produced by fk is 0 because the angle between r and fk is 180˚. The torque produced by N is 0 because for it, r = 0. Now here is a subtler point. Consider the torque produced by the gravitational force. It is:



where r is the blue line connecting O and and the c.g., where Mg is applied, and θ is the angle between r and Mg. But we have a problem, we don’t know how high the c.g. is. So we don’t know what r is. And we don’t know what θ is. But we can circumvent this problem; observe that we can write the torque as:



and observe that from the triangle between r, F, and half the base of the lamp, that *rsinθ* is simply 12cm. So we have,



Now for the torque produced by the force, Fp. It is:



and observe similarly to before that rsinθ for Fp is simply x. So we have,

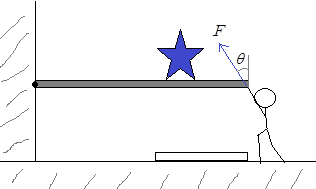


so all total,

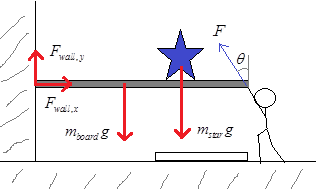


So we can push on the lamp a maximum distance of 60cm above the base of the lamp.

6a. A star sculpture was being held up by a board connected to a wall at one end, and supported by a stand on the other. But an obnoxious 8th grader has kicked out this stand supporting the priceless piece of art. Rushing to the rescue, you exert the force F shown to hold it up. We want to know what this force is. To that end, draw the forces exerted on the board, i.e. those due to gravity on the board, the weight of the sculpture and the horizontal and vertical forces the pin (that holds the board to the wall) exerts.



So we have:



6b. Use N2L for rotation to calculate F. You should place the axis of rotation to be where the pin is. You can take the mass of the board to be m = 5kg, its length to be ℓ = 3m, the mass of the star sculpture to be 15kg, and its distance from the wall to be 2m. You can also take θ = 25°.



6c. Use N2L in the x and y directions to calculte the horizontal and vertical forces the pin exerts on the board.

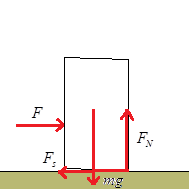
We have:



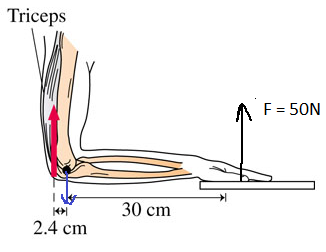
and in the y-direction we have:



7a. Suppose you want to slide a box across the floor, as shown below. So you apply a force F. As a consequence of you trying to slide the box across the floor, a static friction force will oppose this ‘motion’. Draw this force as well as the other two forces acting on the box.



**Question 3**. You push down on a table top with a force F = 50N. What force does your tricep muscle exert on your lower arm.

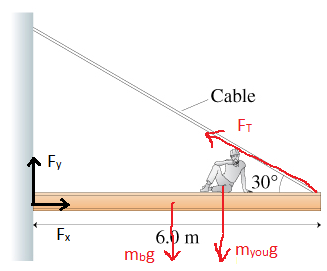




7b. To move the box, your force must be equal to Fsmax = μsFN. But where should you apply your force to keep the box from tipping over? (Hint: place your axis of rotation at the lower right hand corner).



**Question 5**. Suppose you (m = 100kg) are sitting on a beam (m = 1200kg), 4m from the edge. What are the forces Fx, and Fy, that the *wall* exerts on beam in the x and y directions respectively?



Let’s set the axis of rotation to be the joint between the wall and the beam. Then applying the torque equation we can get the tension in the cable…



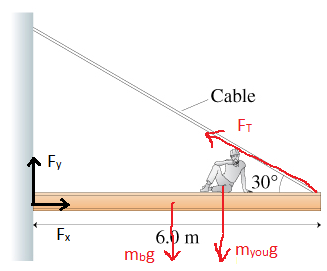
Now use N2L in x direction to get Fx:



and then in the y-direction,



**Question 5**. Suppose you (m = 85kg) are sitting on a beam (m = 900kg), 4.5m from the edge. What are the forces Fx, and Fy, that the *wall* exerts on beam in the x and y directions respectively?



Let’s set the axis of rotation to be the joint between the wall and the beam. Then applying the torque equation we can get the tension in the cable…



Now use N2L in x direction to get Fx:

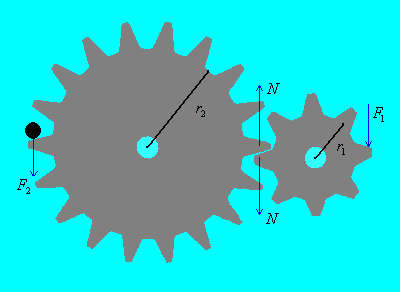


and then in the y-direction,



**Example: Torque amplification through gears**

Consider two gears connected to each other. Let gear 1 have radius r1 and gear 2 have radius r2. If we exert a torque τ1 from force F1 on gear 1, what torque, τ2 will gear 2 exert (on the black pin to its left)? Will it be smaller, larger or the same as τ1?



To answer the question we’ll draw all the forces acting on our gears. F1 will be acting on gear 1. At the contact between the two gears there will be a normal force, N. Gear 1 will exert this force on gear 2 and by N3L, gear 2 will exert the same force on gear 1. Finally, gear 2 will exert a force F2 on the pin, which by N3L, will exert the same force on gear 2.

To determine the torque due to F2, we’ll use N2L (rotation) for both gears.



Now let’s apply N2L (rotation) to gear 2,

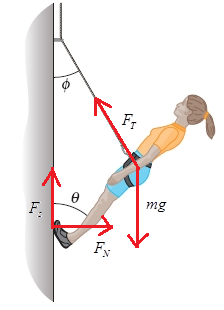


Now filling in what N is we have,



Note how this says that the torque exerted on gear 1 is magnified by gear 2 in proportion to how much larger gear 2 is. So if gear 2 is larger, τ2 will be larger. Conversely, if we make gear 1 smaller (i.e. r1 smaller), then τ2 will get larger. This is why you switch to first gear when climbing hills with your bike, or car.

6. In the figure, a climber with a weight of 480 N is held by a belay rope connected to her climbing harness and belay device; the force of the rope on her has a line of action through her center of mass. The indicated angles are *θ* = 50˚ and *φ* = 40˚. If her feet are on the verge of sliding on the vertical wall, what is the coefficient of static friction between her climbing shoes and the wall?



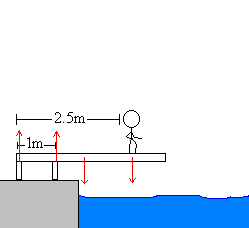
Let ℓ be the distance between her center of mass and her toes. Then the torque equation about this point says,



when on the verge of slipping Fs is at its maximum value Fs = μsFN. So then,



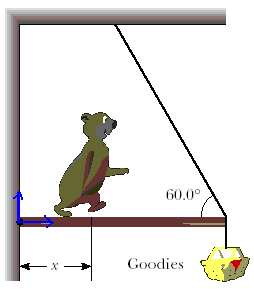
9. A person stands on a 3m long diving board, illustrated below. If the person has a mass of 55 kg, and the diving board has a mass of 120 kg, what is the force exerted by the support on the left?



The forces are shown. Calculating the torque about the right support we have:



10. A hungry 1200N bear walks out on a beam in an attempt to retrieve some "goodies" hanging at the end. The beam is uniform, weighs 610 N, and is 8 m long; the goodies weigh 200 N. Find the horizontal and vertical components (displayed on the figure) of the force the hinge exerts on the beam when the bear is 5m (*x = 5m*) from the edge.



Take the axis of rotation of the beam to be at the hinge first. Then according to N2L we must have:



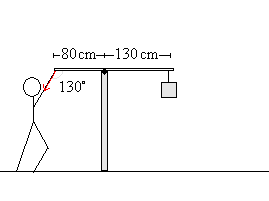
Now applying N2L in the x direction we have:

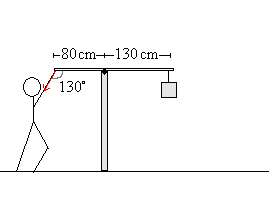


and applying it in the y direction we have:



1. Consider the following weight machine. What force must the person exert at the angle shown, in Newtons, to hold the 40kg mass horizontal? You may take the mass of the bar itself to be 0kg.

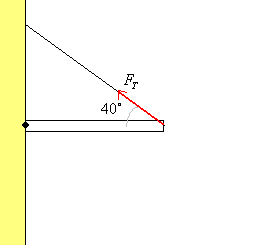




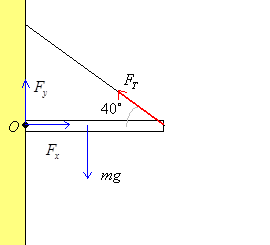
Adding up the torques and setting to zero,



2. Calculate the tension FT in the wire supporting the 27 kg beam, shown below. Note that the wall will exert both a horizontal and vertical force on the beam. So also calculate these forces.



We’ll draw all the forces.



If we take the axis of rotation to be where the pin is, and then add up the torques about O, we get,



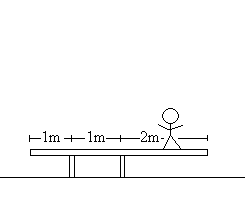
Now since we also want the wall’s forces we will use N2L in the x and y directions…



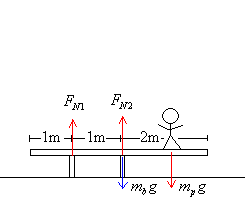
and the other can be obtained from



3. What is the force exerted by the left support attached to the diving board? Suppose the person is 1m away from the right end of the diving board. Also take the mass of the person to be 65kg, and the mass of the diving board to be 50kg. You can take the forces exerted by the supports to be purely vertical.



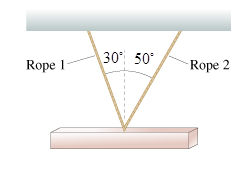
Labelling all the forces



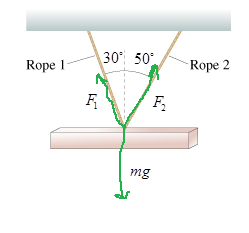
And now let’s apply the torque equation about the second support – this is the easiest way to do this.



**Question 3.** Consider the beam being held by the ropes below. What is the tension in both ropes if the beam has a mass m = 225kg?



Forces look like this:



N2L in x direction yields:



and in y-direction:



Plugging in x-equation:



and then for F1 we have:

